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Inverse Problem in Classical Mechanics: Dissipative Systems

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We show how the ambiguity of Lagrangian and Hamiltonian descriptions for conservative systems gives rise to an analogous ambiguity for dissipative systems. For a subclass of them we also give a Lagrangian description.

1. INTRODUCTION

If we adopt the point of view that in classical particle mechanics a system is completely specified by the equations of motion, then a given system may admit many different Lagrangian descriptions or none. These considerations are by now well known in the literature (Havas, 1957; Currie and Saletan, 1966; Gelman and Saletan, 1973; Caratù et al., 1976; Marmo and Saletan, 1977; Dodonov et al., 1978; Sarlet and Cantrijn, 1978; Okubo, 1980; Sarlet and Bahar, 1980; Sarlet, Engels, and Bahar, 1980; Dodonov et al., 1981; Crampin, 1981; Kocik, 1981; Sarlet, 1981). A recent comprehensive discussion of the so-called Helmholtz conditions for a second-order system to admit a Lagrangian description is given by Sarlet (1981). Different Lagrangian descriptions of the same system give rise to different "energies," i.e., $E_{\mathfrak{L}} = \dot{q}^i \partial \mathfrak{L} / \partial \dot{q}^i - \mathfrak{L}$. It turns out that whether a dynamical system is dissipative or not may depend on its Lagrangian description.

Several dissipative systems are known to admit a Lagrangian description. In this paper we show the geometrical origin of such Lagrangians in the context of global differential geometry.

The organization of the paper is the following. In Section 2 we review in a simple way the inverse problem and some well known examples of dissipative systems in the language of differential forms. In Section 3 we show a general procedure to give a Lagrangian description for a class of dissipative systems, while Section 4 is devoted to considering the ambiguity in the Lagrangian description to cover different dissipative forces. In Section 5 we touch upon Noether's theorem in such a setting. A recent analysis of vector fields generating invariants for dissipative systems is given by Cantrijn (1981). Section 6 deals with conclusions and some open problems.

2. PRELIMINARIES

2.1. Inverse Problem: A Simple Review. On the configuration space Q, with coordinates $(q) = (q_1, \ldots, q_n)$, Newtonian equations in normal form are

$$\ddot{q}_i = G_i(q_k, \dot{q}_k), \qquad i \in \{1, \ldots, n\}, \qquad k \in \{1, \ldots, n\} \tag{1}$$

A Lagrangian description is given in terms of $\mathfrak{L} = \mathfrak{L}(q_k, \dot{q}_k)$ through Euler–Lagrange equations:

$$\frac{d}{dt} \frac{\partial \mathfrak{L}}{\partial \dot{q}^i} - \frac{\partial \mathfrak{L}}{\partial q^i} = 0, \qquad i \in \{1, \ldots, n\} \tag{2}$$

The inverse problem, as presented in the literature (see Santilli, 1978, for an extensive treatment; see also Havas, 1957; Currie and Saletan, 1966; Gelman and Saletan, 1973; Caratù et al., 1976; Marmo and Saletan, 1977; Dodonov et al., 1978; Dodonov et al., 1981; Okubo, 1980; Sarlet and Cantrijn, 1978; Cantrijn and Sarlet, 1981; Sarlet and Bahar, 1980; Sarlet, Engels and Bahar, 1980; Crampin, 1981; Kocik, 1981; Sarlet, 1981), starts from the remark that if $a_{ij}(t, q, \dot{q})$ is any nondegenerate matrix, equations (1) and

$$a_{ij}(\ddot{q}_i - G_i) \equiv 0 \tag{3}$$

are completely equivalent (by this we mean that they give rise to the same set of trajectories). Then the Lagrangian description is posed for equations (3), where a_{ij}'s are unknown. In the language of differential forms, the problem amounts to finding integrating factors for some 2-forms associated to equations (1). This association goes back to Cartan (Cartan, 1922; Gallissot, 1951; Gallissot, 1952).

In terms of differential forms, the following is a simple presentation of the problem, but see also Sarlet and Cantrijn (1978), Cantrijn and Sarlet (1981), Sarlet and Bahar (1980), Sarlet, Engels, and Bahar (1980), Crampin (1981), Kocik (1981), Sarlet (1981). (For different approaches see Tonti, 1969; Takens, 1979). On $TQ\times R$ (Abraham and Marsden, 1978; Caratù et al., 1976) equations (1) are reduced to

$$\frac{dq_i}{dt}=\dot{q}_i$$
$$\frac{d\dot{q}_i}{dt}=G_i \qquad i\in\{1,\ldots,n\} \tag{4}$$
$$\frac{dt}{dt}=1$$

In collective coordinates (ξ,t) (this allows for other carrier spaces) we have

$$\begin{aligned}\frac{d\xi^i}{dt}&=\Delta^i\\ \frac{dt}{dt}&=1\end{aligned}\qquad i\in\{1,\ldots,2n\} \tag{5}$$

Such equations are described by the vector field $\Delta(\xi,t)=\Delta^i\partial/\partial\xi^i+\partial/\partial t$, or by the set of 1-forms $\gamma^i=d\xi^i-\Delta^i\,dt, i\in\{1,\ldots,2n\}$. The remarkable property of the γ's is

$$i_\Delta\gamma^i=0, \qquad \forall i\in\{1,\ldots,2n\}$$
$$i_\Delta\,dt=1 \tag{6}$$

An independent set of $2n+1$ forms determines (and is determined by) a vector field Δ by equations (6).

By considering combinations $\omega=a_{ij}\gamma^i\wedge\gamma^j, \|a_{ij}\|$ nondegenerate, Δ can be determined as the unique vector field such that $i_\Delta\omega=0$, $i_\Delta\,dt=1$. It is quite obvious that Δ does not determine a unique ω, and different nondegenerate matrices will give rise to different ω's. The additional requirement of closure for ω, i.e., $d\omega = 0$, restricts the set of allowed a_{ij}'s, but still leaving room for many of them.

As a matter of fact, for any point m of the carrier space, such that $\Delta(m) \neq 0$, it is possible to find a neighborhood U on which there is an infinite set of closed ω's. By using the "straightening out theorem" [see Abraham and Marsden (1978), p. 67, Theorem 2-1-9], we can find a coordinate system for U, say $(\eta) = (\eta^1, \eta^2, \ldots, \eta^{2n}, t)$, such that

$$\Delta = \frac{\partial}{\partial \eta^1} + \frac{\partial}{\partial t}$$

and

$$\gamma^1 = d\eta^1 - 1\, dt, \qquad \gamma^2 = d\eta^2, \ldots, \gamma^{2n} = d\eta^{2n}$$

$$\omega = a_{ij} \gamma^i \wedge \gamma^j, \qquad i_\Delta \omega = 0, \qquad i_\Delta\, dt = 1, \qquad a_{ij} = -a_{ji}$$

The expression of $\omega = a_{ij} d\eta^i \wedge d\eta^j - a_{1j} dt \wedge d\eta^j$ shows that in the time-independent case $H_{\|a_{ij}\|} = a_{1j}\eta^j$ will give a possible Hamiltonian for the symplectic structure $a_{ij} d\eta^i \wedge d\eta^j$. Thus, *locally the "inverse problem" at the symplectic level has always infinite solutions and any constant of the motion which has regular values in U can be a possible Hamiltonian in U* [see Abraham and Marsden (1978) for the definition of regular value].

Of course, not all of such 2-forms can be given a Lagrangian description if we only allow for transformations which are Q-transformations (Marmo and Saletan, 1977).

For the Lagrangian description we go back to coordinates $(q^i, \dot{q}^i, t)$. The dynamics is

$$\Delta \equiv \begin{cases} \dfrac{dq^i}{dt} = \dot{q}^i \\ \dfrac{d\dot{q}^i}{dt} = \Delta^i \\ \dfrac{dt}{dt} = 1 \end{cases}$$

with associated 1-forms:

$$\alpha^i = dq^i - \dot{q}^i\, dt, \qquad \beta^i = d\dot{q}^i - \Delta^i\, dt$$

Allowed 2-forms are

$$a_{ij}\alpha^i \wedge \beta^j + b_{ij}\alpha^i \wedge \alpha^j + c_{ij}\beta^i \wedge \beta^j$$

On $TQ \times R$, closed 2-forms are described by a Lagrangian function $\mathfrak{L}$ if

$$\omega = d\theta_{\mathfrak{L}} - dE_{\mathfrak{L}} \wedge dt$$

where

$$\theta_{\mathfrak{L}} = \frac{\partial \mathfrak{L}}{\partial \dot{q}^i} dq^i, \qquad E_{\mathfrak{L}} = \dot{q}^i \frac{\partial \mathfrak{L}}{\partial \dot{q}^i} - \mathfrak{L}$$

in a natural chart $(q, \dot{q}, t)$. Thus, a necessary condition for a Lagrangian description of allowed ω's is $c_{ij} \equiv 0$.

Now we are not permitted to make use of the "straightening out theorem," for the previous η's coordinates would mix up q's and $\dot{q}$'s and destroy the vectorial nature of fibers in TQ. For a necessary and sufficient condition for a Lagrangian description in the time-independent case see Balachandran et al. (1980). For a recent comprehensive discussion see Sarlet (1981).

2.2· Examples. *Example* 1. On $Q \equiv R$, $TQ = R^2$ consider the dynamical system:

$$\Delta \equiv \begin{cases} \dfrac{dq}{dt} = \dot{q} \\ \dfrac{d\dot{q}}{dt} = -\gamma\dot{q} \\ \dfrac{dt}{dt} = 1 \end{cases} \qquad \text{or } \ddot{q} + \gamma\dot{q} = 0$$

From the above considerations, we get $\theta_{\mathfrak{L}} = \ln \dot{q}\, dq$, and

$$\mathfrak{L} = \dot{q}(\ln \dot{q} - 1) - \gamma q, \qquad E_{\mathfrak{L}} = \dot{q} + \gamma q$$

This example shows that, by considering $\ddot{q} = 0$, we get $\mathfrak{L}_1 = \frac{1}{2}\dot{q}^2$, $E_{\mathfrak{L}_1} = \frac{1}{2}\dot{q}^2$ and $\ddot{q} + \gamma\dot{q} = 0$ might be thought of as a dissipative system with respect to $\omega_{\mathfrak{L}_1} = d_{\dot{q}} \wedge d_q$, for $i_\Delta \omega_{\mathfrak{L}_1} = \dot{q}\, d\dot{q} - \gamma\dot{q}\, dq$ and $(d/dt)E_{\mathfrak{L}_1} = -\gamma\dot{q}\, dq(\Delta) = -\gamma\dot{q}^2$. On the other hand, $\omega_{\mathfrak{L}} = (1/\dot{q})\, dq \wedge d\dot{q}$ gives $i_\Delta \omega_{\mathfrak{L}} = d\dot{q} + \gamma\, dq$ and $(d/dt)E_{\mathfrak{L}} \equiv 0$ (even if we have to restrict ourselves to $TQ - \{\text{zero section}\}$, this does not create troubles for the zero section is an invariant set of our dynamics).

So, a given dynamical system can be "dissipative" with respect to a given $\omega_{\mathfrak{L}_1}$ and "conservative" with respect to some other.

Example 2. The damped harmonic oscillator.

On $TQ = R \times R$, consider

$$\Delta \equiv \begin{cases} \dfrac{dq}{dt} - \dot{q} = 0 \\ \dfrac{d\dot{q}}{dt} + \gamma\dot{q} + \omega^2 q = 0, \qquad \omega^2 - \dfrac{\gamma^2}{4} > 0 \end{cases}$$

The associated system $\alpha = dq - \dot{q}\,dt$, $\beta = d\dot{q} + (\gamma\dot{q} + \omega^2 q)\,dt$ allows for a Lagrangian description given by (Havas, 1957)

$$\mathfrak{L} = \frac{2\dot{q} + \gamma q}{2q(\omega^2 - \gamma^2/4)^{1/2}} \tan^{-1}\left[\frac{2\dot{q} + \gamma q}{2q(\omega^2 - \gamma^2/4)^{1/2}}\right] - \tfrac{1}{2}\ln|\dot{q}^2 + \gamma q\dot{q} + \omega^2 q^2|$$

Here $q = 0$ is not an invariant set, thus $\mathfrak{L}$ fails to give a global Lagrangian for Δ.

Both systems we have discussed above can be given a global Lagrangian description if we allow for time-dependent Lagrangians. The procedure through which we construct a Lagrangian is pretty general, so we review briefly time-dependent Lagrangian formalism in terms of differential geometry.

3. LAGRANGIAN DESCRIPTION

3.1. Time-Dependent Lagrangian Formalism (Abraham and Marsden, 1978). We introduce here some notations. On $TQ \times R$ an extended second-order vector field Δ is expressed in local terms by

$$\Delta = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$$

We say that Δ *has a Lagrangian description if* $i_\Delta \omega_{\mathfrak{L}} = 0$ *and* $i_\Delta\, dt = 1$, where $\omega_{\mathfrak{L}}$ is defined by $\omega_{\mathfrak{L}} = d(d_v \mathfrak{L})$. We recall that (Godbillion, 1969; Marmo and Saletan, 1977) a "vertical operator" v is defined by

$$X \in \mathfrak{X}(TQ), \qquad v: X = a_i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial \dot{q}^i} \mapsto vX = a_i \frac{\partial}{\partial \dot{q}^i}$$

with adjoint v_* defined by

$$\theta \in \mathfrak{X}^*(TQ), \qquad v_*: \theta = a_i\, dq^i + b_i\, d\dot{q}^i \mapsto v_*\theta = b_i\, dq^i$$
$$f \in \mathfrak{F}(TQ), \qquad v_*: f \mapsto v_* f = f$$

v_* defines a differential operator d_v on functions by setting $v_*(df) = d_v f$. (For extension of d_v to forms see Godbillon, 1969.) In local coordinates

$$d_v \mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \dot{q}^i} dq^i, \qquad E_\mathfrak{L} = i_\Delta d_v \mathfrak{L} - \mathfrak{L}$$

for some second-order vector field Δ.

Then, if $\theta_\mathfrak{L} = d_v \mathfrak{L} - E_\mathfrak{L}\, dt$, it is $\omega_\mathfrak{L} = dd_v \mathfrak{L} - dE_\mathfrak{L} \wedge dt = d\theta_\mathfrak{L}$, and $\Delta = \partial/\partial t + \Delta_0$ satisfies

$$i_\Delta \omega_\mathfrak{L} = 0, \qquad i_\Delta\, dt = 1 \tag{7}$$

3.2. Dissipative Systems (Shahshahani, 1972; Cantrijn, 1981). On TQ, let Δ be a second-order vector field which has a Lagrangian description in terms of $\mathfrak{L} \in \mathfrak{F}(TQ)$, by

$$i_\Delta \omega_\mathfrak{L} = -\, dE_\mathfrak{L}$$

On TQ basic 1-forms are defined as being generated by $\pi^*(\mathfrak{X}^*(Q))$; i.e., in local coordinates they are of the form

$$\alpha = \alpha(q, \dot{q}) = \sum_{i=1}^{n} a_i(q, \dot{q})\, dq^i$$

Forces can be defined as vertical vector fields, i.e., $Y \in \mathfrak{X}(TQ)$ such that $vY = 0$.

For a given Lagrangian $\mathfrak{L}$, we can use the pairing

$$i_Y \omega_\mathfrak{L} = \alpha_y$$

to associate forces and basic 1-forms on TQ. Forces F, associated to closed basic 1-forms, can be implemented in the vector field Δ, as $\Delta_F = \Delta + F$, in such a way that $\Delta + F$ has a Lagrangian description. As a matter of fact, if $i_F \omega_\mathfrak{L} = df$, $i_\Delta \omega_\mathfrak{L} = -\, dE_\mathfrak{L}$, the vector field $\Delta_F = \Delta + F$ has a Lagrangian description in terms of $\mathfrak{L} + f$ (Marmo and Saletan, 1977).

We can define dissipative forces with respect to a given $\mathfrak{L}$ as those vertical vector fields which are associated to nonclosed basic 1-forms on TQ; i.e., $i_F\omega_{\mathfrak{L}} = \alpha$, α basic $d\alpha \neq 0$.[1]

A subclass of dissipative forces with respect to $\mathfrak{L}$ is given by those basic 1-forms α which satisfy

$$d_v\alpha = 0$$

For these 1-forms it is possible to find a function $\mathfrak{F} \in \mathcal{K}(TQ)$ such that $\alpha = d_v\mathfrak{F} - \mathfrak{F} \in \mathcal{K}(TQ)$ is a generalization of the "Rayleigh's dissipation function" (Goldstein, 1980).

Each dynamical system $\Delta_F = \Delta + F$ of this subclass needs two functions, $\mathfrak{L}$ and $\mathfrak{F}$, to be determined as

$$i_{\Delta_F}\omega_{\mathfrak{L}} = -dE_{\mathfrak{L}} + d_v\mathfrak{F}$$

An example of such a situation is given by van der Pol's equation (van der Pol, 1922; 1926):

$$\ddot{q} + \mu(q^2 - 1)\dot{q} + q = 0$$

Previous algorithm gives:

$$\mathfrak{L} = \tfrac{1}{2}(\dot{q}^2 - q^2), \qquad \mathfrak{F} = \tfrac{1}{2}\mu(q^2 - 1)\dot{q}^2$$

$$\Delta_F = \dot{q}\frac{\partial}{\partial q} + [-\mu(q^2 - 1)\dot{q} + q]\frac{\partial}{\partial \dot{q}}$$

$$i_{\Delta_F}\omega_{\mathfrak{L}} = -dE_{\mathfrak{L}} + d_v\mathfrak{F}$$

[1] The usual notion of dissipative systems starts from

$$\frac{d}{dt}\frac{\partial\mathfrak{L}}{\partial\dot{q}^i} - \frac{\partial\mathfrak{L}}{\partial q^i} = Q_i$$

where Q_i do not admit a Lagrangian description. In coordinate-free notation we have

$$L_\Delta\theta_{\mathfrak{L}} - d\mathfrak{L} = \alpha, \qquad \alpha = Q_i\,dq^i$$

by using Cartan's identity $L_\Delta = i_\Delta d + di_\Delta$, we get $i_\Delta\omega_{\mathfrak{L}} = d(\mathfrak{L} - i_\Delta\theta_{\mathfrak{L}}) + \alpha$. This relation suggests our use of dissipative. On the other hand, $i_\Delta(L_\Delta\theta_{\mathfrak{L}} - d\mathfrak{L}) = i_\Delta\alpha$ gives

$$L_\Delta(i_\Delta\theta_{\mathfrak{L}} - \mathfrak{L}) = L_\Delta(E_{\mathfrak{L}}) = -i_\Delta\alpha \qquad (*)$$

Thus what we are considering here as "dissipative" are second-order systems which do not preserve Poisson brackets on the tangent bundle. If we stick to $(*)$ then $\alpha = F_{ij}\dot{q}^i dq^j$, $F_{ij} = -F_{ji}$, would give $i_\Delta\alpha = F_{ij}\dot{q}^i\dot{q}^j = 0$; thus dynamical evolution would preserve $E_{\mathfrak{L}}$ while not preserving Poisson brackets.

We have

$$0 = (-dE_{\mathfrak{L}} + d_v\mathfrak{F})(\Delta_F) = -dE_{\mathfrak{L}}(F) + d_v\mathfrak{F}(\Delta)$$

$$\frac{dE_{\mathfrak{L}}}{dt} = L_Y\mathfrak{F}$$

where $Y = \dot{q}^i\,\partial/\partial\dot{q}^i$ is the so-called Liouville vector field (Goldbillon, 1969).

We will investigate in some details a subclass of such systems, namely, those for which $dd_v\mathfrak{F} = \gamma dd_v\mathfrak{L}, \gamma \in \mathbb{R}$. They allow for a time-dependent Lagrangian description in terms of $\mathfrak{L}' = e^{-\gamma t}\mathfrak{L}$.

We notice that $dd_v\mathfrak{F} = \gamma dd_v\mathfrak{L}$ implies $d_v\mathfrak{F} = \gamma d_v\mathfrak{L} + \alpha$, where α is a closed basic 1-form. Such an α can be implemented, at least locally, in terms of $\mathfrak{L} + f$, where $df = \alpha$. Thus we restrict our attention to $\mathfrak{F}$'s which satisfy $d_v\mathfrak{F} = \gamma d_v\mathfrak{L}$.

In this case, defining $\mathfrak{L}' = e^{-\gamma t}\mathfrak{L}$, we get

$$\omega_{\mathfrak{L}'} = e^{-\gamma t}\omega_{\mathfrak{L}} - e^{-\gamma t}\gamma dt \wedge d_v\mathfrak{L} - e^{-\gamma t} dE_{\mathfrak{L}} \wedge dt$$

$$= e^{-\gamma t}\left[\omega_{\mathfrak{L}} + (d_v\mathfrak{F} - dE_{\mathfrak{L}}) \wedge dt\right]$$

Equations $i_{\tilde{\Delta}_F}\omega_{\mathfrak{L}'} = 0$, $i_{\tilde{\Delta}_F}dt = 1$ are solved by $\tilde{\Delta}_F = \Delta_F + \partial/\partial t$, where $i_{\tilde{\Delta}_F}\omega_{\mathfrak{L}} = -dE_{\mathfrak{L}} + d_v\mathfrak{F}$, for

$$i_{\tilde{\Delta}_F}\omega_{\mathfrak{L}'} = e^{-\gamma t}\left[i_{\Delta_F}\omega_{\mathfrak{L}} + (d_v\mathfrak{F} - dE_{\mathfrak{L}})(\Delta_F)\, dt - d_v\mathfrak{F} + dE_{\mathfrak{L}}\right]$$

$$= e^{-\gamma t}(d_v\mathfrak{F} - dE_{\mathfrak{L}} - d_vF + dE_{\mathfrak{L}}) = 0$$

where we have used

$$i_{\Delta_F} i_{\Delta_F}\omega_{\mathfrak{L}} = (d_v\mathfrak{F} - dE_{\mathfrak{L}})(\Delta_F) = 0$$

Going back to the previous examples we have the following:

Example 1. For the dynamical system described by

$$\ddot{q} + \gamma\dot{q} = 0$$

we get a possible time-dependent Lagrangian description in terms of $\mathfrak{L}' = \frac{1}{2}e^{\gamma t}\dot{q}^2$, for

$$i_{\gamma\dot{q}\frac{\partial}{\partial\dot{q}}}(d\dot{q} \wedge dq) = \gamma\dot{q}\, dq = \gamma d_v\left(\tfrac{1}{2}\dot{q}^2\right)$$

Example 2. For the damped harmonic oscillator, described by

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0$$

again

$$i_{\gamma\dot{q}\partial/\partial\dot{q}}(d\dot{q} \wedge dq) = \gamma\dot{q}\, dq = \gamma d_v(\tfrac{1}{2}\dot{q}^2)$$

implies a Lagrangian description in terms of

$$e^{\gamma t}(\tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}\omega^2 q^2)$$

It is possible to generalize our previous procedure to the case of time-dependent dissipative forces replacing $\gamma\epsilon R$ with $\gamma = \gamma(t)$. This situation is the same given by Lane–Emden equation (Cantrijn and Sarlet, 1981):

$$\ddot{q} + \frac{2}{t}\dot{q} + q^5 = 0$$

i.e., Poisson equation for a spherically symmetric system in hydrostatic equilibrium (Chandrasekhar, 1958). Since we have

$$i_{(2/t)\dot{q}\partial/\partial\dot{q}}(d\dot{q} \wedge dq) = \frac{2}{t}\dot{q}\, dq = \frac{2}{t} d_v(\tfrac{1}{2}\dot{q}^2)$$

a possible Lagrangian description is given by (Logan, 1977)

$$\mathcal{L}' = e^{2\ln t}(\tfrac{1}{2}\dot{q}^2 - \tfrac{1}{6}q^6)$$

At the Hamiltonian level a Hamiltonian dynamics is defined by

$$i_\Delta\, d\theta = -dH$$

Basic 1-forms on T^*Q are given, in local terms, as $\alpha = a_i(q, p)\, dq^i$ and vector fields associated to basic 1-forms are given by $i_F d\theta = \alpha$. The local expression for F will be $F = a_i\, \partial/\partial p_i$.

Hamilton's equations for Δ are

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

and those associated to $\Delta_F = \Delta + F$ will be

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + a_i$$

Again, if $d\alpha = 0$, i.e., locally $\alpha = df$, we get

$$i_{\Delta_F} d\theta = -d(H - f) \quad \text{and} \quad a_i = -\frac{\partial f}{\partial q^i}$$

On the phase space where we allow for general canonical transformations the "fibered" nature of T^*Q gets lost and the notion of basic 1-form is not preserved. Nevertheless it is still possible to consider the analog of $d_v \mathcal{F} = \gamma d_v \mathfrak{L}$ by considering $\alpha = \gamma\theta$. The ambiguity is choosing θ out of $d\theta$ is given by a closed 1-form. Such closed 1-form can be taken in account, at least locally, through $H' = H - f$, where $\alpha = df$. Thus we are left with

$$i_F d\theta = \gamma\theta$$

Dynamical systems as $\Delta_F = \Delta + F$ can be given a time-dependent Hamiltonian description through

$$\tilde{\theta} = e^{\gamma t}\theta$$

Equations (7) are replaced by

$$i_{\tilde{\Delta}_F} d\left[e^{\gamma t}(\theta - H\,dt)\right] = 0, \qquad i_{\tilde{\Delta}_F} dt = 1$$

A remark is in order here. The vector field F determined by $i_F d\theta = \gamma\theta$ may turn out to be Hamiltonian with respect to some other symplectic structure.

The following example illustrates the situation.

An unusual Lagrangian description for the free particle, $\mathfrak{L} = e^{\dot{q}}$, on $TQ = R \times R$, gives equations of motion $e^{\dot{q}}\ddot{q} = 0 \Leftrightarrow \ddot{q} = 0$. The "dissipative force" such that $i_F dd_v \mathfrak{L} = d_v \mathfrak{L}$ is given by $F = 1\,\partial/\partial\dot{q}$, for $d_v \mathfrak{L} = e^{\dot{q}}\,dq$, $dd_v \mathfrak{L} = e^{\dot{q}} d\dot{q} \wedge dq$. Of course, the vector field $\Delta = \dot{q}\,\partial/\partial q$, by adding F, becomes

$$\Delta_F = \dot{q}\frac{\partial}{\partial q} + 1\frac{\partial}{\partial \dot{q}}$$

which has an obvious time-dependent Lagrangian description through $\mathfrak{L} = \frac{1}{2}\dot{q}^2 - q$, without going to $\mathfrak{L}' = e^{\gamma t + \dot{q}}$ $(\gamma = 1)$.

4. DIFFERENT LAGRANGIAN DESCRIPTIONS AND DISSIPATIVE FORCES

By using different Lagrangian descriptions, is it possible to obtain different "dissipative forces" by the procedure $i_F\omega_{\mathcal{L}} = d_v\mathcal{L}$? This example shows that it may not be the case. For instance, let $\mathcal{L}_1 = \dot{q}_1\dot{q}_2 - q_1q_2$ and $\mathcal{L}_2 = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - q_2^2)$ be two different Lagrangians for the isotropic harmonic oscillator with two degrees of freedom. It turns out

$$\omega_{\mathcal{L}_1} = d\dot{q}_1 \wedge dq_2 + d\dot{q}_2 \wedge dq_1, \qquad \theta_{\mathcal{L}_1} = \dot{q}_1\, dq_2 + \dot{q}_2\, dq_1$$

$$\omega_{\mathcal{L}_2} = d\dot{q}_1 \wedge dq_1 + d\dot{q}_2 \wedge dq_2, \qquad \theta_{\mathcal{L}_2} = \dot{q}_1\, dq_1 + \dot{q}_2\, dq_2$$

and $i_{F_1}\omega_{\mathcal{L}_1} = \theta_{\mathcal{L}_1}$, $i_{F_2}\omega_{\mathcal{L}_2} = \theta_{\mathcal{L}_2}$ give

$$F_1 = F_2 = \dot{q}_1\frac{\partial}{\partial\dot{q}_1} + \dot{q}_2\frac{\partial}{\partial\dot{q}_2}$$

How should we choose different Lagrangian descriptions to get different F's? First, notice that, in general

$$i_{F_i}d\theta_{\mathcal{L}_i} = \theta_{\mathcal{L}_i} \Rightarrow n\theta_{\mathcal{L}_i} \wedge \left(d\theta_{\mathcal{L}_i}\right)^{n-1} = i_{F_i}\left(d\theta_{\mathcal{L}_i}\right)^n \qquad (i = 1,2)$$

We can write

$$\left(d\theta_{L_1}\right)^n = f_{12}\left(d\theta_{\mathcal{L}_2}\right)^n$$

with $f_{12} \in \mathcal{K}(TQ)$. If we take $F_1 = F_2$, we get

$$f_{12}\theta_{\mathcal{L}_2} \wedge \left(d\theta_{\mathcal{L}_2}\right)^{n-1} = \theta_{\mathcal{L}_1} \wedge \left(d\theta_{\mathcal{L}_1}\right)^{n-1}$$

This is a necessary and sufficient condition for $F_1 = F_2$. As a matter of fact, any volume element Ω defines an isomorphism

$$\mathfrak{X} \mapsto \Lambda^{2n-1}$$

by $X \to i_X\Omega$.

Going back to the example of the harmonic oscillator, we get

$$\omega_{\mathcal{L}_2} \wedge \omega_{\mathcal{L}_2} = \omega_{\mathcal{L}_1} \wedge \omega_{\mathcal{L}_1} \Rightarrow f_{12} = 1$$

and

$$\theta_{\mathfrak{L}_1} \wedge d\theta_{\mathfrak{L}_1} = \dot{q}_1 dq_2 \wedge d\dot{q}_2 \wedge dq_1 + \dot{q}_2 dq_1 \wedge d\dot{q}_1 \wedge dq_2$$

$$\theta_{\mathfrak{L}_2} \wedge d\theta_{\mathfrak{L}_2} = \dot{q}_1 dq_1 \wedge d\dot{q}_2 \wedge dq_2 + \dot{q}_2 dq_2 \wedge d\dot{q}_1 \wedge dq_1$$

Thus, conditions for $F_1 = F_2$ are satisfied.

Comments: Even if we find different Lagrangians satisfying previous conditions for $F_1 \neq F_2$, we are not assured that we can recover all possible dissipative forces.

5. NOETHER'S THEOREM FOR SOME DISSIPATIVE SYSTEMS (Cantrijn, 1981)

In this section we would like to consider Noether's theorem in its simplest form, i.e., in terms of point transformations which preserve the Lagrangian.

On TQ, if $i_\Delta \omega_{\mathfrak{L}} = -dE_{\mathfrak{L}1} X \in \mathfrak{X}(Q)$, and $\dot{X}$ is the lifting of X to TQ, then

$$L_{\dot{X}}\mathfrak{L} = 0 \Rightarrow L_{\dot{X}}\Delta = 0, \qquad L_{\dot{X}}E_{\mathfrak{L}} = 0$$

and

$$L_{\dot{X}}d_v = d_v L_{\dot{X}}$$

(see Marmo and Saletan, 1977). Noether's theorem asserts that $i_{\dot{X}}d_v\mathfrak{L}$ is a constant of the motion. In our formalism this is given by

$$i_{\dot{X}}(L_\Delta d_v\mathfrak{L} - d\mathfrak{L}) = 0 = L_\Delta(i_{\dot{X}}d_v\mathfrak{L})$$

If we build up a dissipative system through $i_{\Delta_F}\omega_{\mathfrak{L}} = -dE_{\mathfrak{L}} + \gamma d_v\mathfrak{L}$ and its time-dependent Lagrangian description:

$$i_{\tilde{\Delta}_F}\omega_{\mathfrak{L}'} = 0, \qquad i_{\tilde{\Delta}_F}dt = 1$$

we have $i_{\dot{X}}d_v(e^{-\gamma t}\mathfrak{L})$ is a constant of the motion for $\tilde{\Delta}_F$, from

$$L_{\tilde{\Delta}_F}\left[i_{\dot{X}}d_v(e^{-\gamma t}\mathfrak{L})\right] = L\tilde{\Delta}_F\left(e^{-\gamma t}i_{\dot{X}}\theta_{\mathfrak{L}}\right)$$

$$= -e^{-\gamma t}\gamma i_{\dot{X}}\theta_L + e^{-\gamma t}\gamma i_{\dot{X}}\theta_{\mathfrak{L}} = 0$$

Example: *The Damped Harmonic Oscillator*. On $TQ = R^4 = R^2 \times R^2$, consider

$$\ddot{q}^i + \gamma \dot{q}^i + \omega^2 q^i = 0, \qquad i \in \{1,2\}$$

with

$$\Delta_F = \dot{q}^i \frac{\partial}{\partial q^i} - (\gamma q^i + \omega^2 q^i) \frac{\partial}{\partial \dot{q}^i} \cdot X_A = a_{ij} q^i \frac{\partial}{\partial q^i}$$

gives

$$\dot{X}_A = a_{ij} q^i \frac{\partial}{\partial q^j} + a_{ij} \dot{q}^i \frac{\partial}{\partial \dot{q}^j}$$

and

$$[X_A, \Delta_F] = a_{ij} \dot{q}^i \frac{\partial}{\partial q^j} - \left(a_{ij} \gamma \dot{q}^i \frac{\partial}{\partial \dot{q}^j} + \omega^2 a_{ij} q^i \frac{\partial}{\partial q^j} \right)$$
$$- a_{ij} \dot{q}^i \frac{\partial}{\partial q^j} + (\gamma \dot{q}^i + \omega^2 q^i) a_{ij} \frac{\partial}{\partial \dot{q}^j} \equiv 0$$

Thus, Δ_F is invariant under the homogeneous linear group $GL(2, R)$. Among equivalent Lagrangian descriptions for our system we can choose the usual one:

$$\mathfrak{L} = \tfrac{1}{2} e^{-\gamma t} \left(\dot{q}^{i^2} - q^{i^2} \right)$$

We have

$$L_{X_A} \left(\dot{q}^{i^2} - \dot{q}^2 \right) = 0$$

for $a_{ij} = -a_{ji}$, $a_{ij} a_{jk} = \delta_{ik}$; i.e., the given Lagrangian selects the rotation group. Thus $e^{-\gamma t} a_{ij} \dot{q}^i q^j = i_{X_A} d_v \mathfrak{L}$ is a constant of the motion. In fact

$$L_{\tilde{\Delta}_F} \left(e^{-\gamma t} a_{ij} \dot{q}^i q^j \right) = 0$$

6. CONCLUSIONS

We have shown in elementary terms how the ambiguity of Lagrangian and Hamiltonian descriptions for conservative systems gives rise to an analogous ambiguity for the description of dissipative systems.

For a subclass of dissipative systems we have given a time-dependent Lagrangian description, explaining its geometrical origin. We leave it as an open problem how to deal with remaining ones. We would like to notice here that any dynamical system on Q can be given a Hamiltonian description by the following procedure.

Let Δ be a dynamical system on Q. On T^*Q we define a unique $\Delta\uparrow$, by setting $L_{\Delta\uparrow}\theta_0 = 0$ (this gives a unique $\Delta\uparrow$, for $i_{\Delta\uparrow}\, d\theta_0 = L_{\Delta\uparrow}\theta_0 - di_{\Delta\uparrow}\theta_0$, and noticing that $i_{\Delta\uparrow}\theta_0$ depends only on Δ). $\Delta\uparrow$ has Hamiltonian $H = i_{\Delta\uparrow}\theta_0$. This means that by adding at most n variables it is always possible to put any system in a Hamiltonian form. It is interesting to ask for a minimal set of variables we need to add to give our system a Hamiltonian form. Our treatment can be thought of as a contribution in characterizing a subclass of systems for which only one variable will do.

We have not touched upon the problem of quantization. A straightforward canonical quantization would give rise to problems for the Poisson brackets of conjugate variables have an exponential time dependence. To deal with some of such systems in Dodonov and Man'ko (1977) the concept of loss-energy states is introduced.

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